

## ON THE SMOOTHNESS OF GENERATORS

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WE WILL investigate the relationship between the smoothness of flows and their generating vector fields, and the analogous problem for foliations. It is well known that the flow (foliation) generated by a vector field (vector distribution, or plane field) is of the same differentiability class. The converse fails, in general; a  $C^r$  flow need not have a  $C^r$  generator. For example, if  $Y$  is a  $C^r$  vector field generating a  $C^r$  flow  $\psi$ , and  $g$  is a  $C^r$  diffeomorphism, nowhere  $C^{r+1}$ , then "conjugation" by  $g$  produces the flow  $\phi_t(p) = g(\psi_t(g^{-1}(p)))$ , whose generator  $X = Dg \circ Y \circ g^{-1}$  need only be  $C^{r-1}$ . We shall show that this example illustrates the most general case: given *any*  $C^r$  flow, one may always construct a  $C^r$  diffeomorphism conjugating that flow to another which is generated by a  $C^r$  vector field.

The existence of generating vector fields, of class  $C^{r-1}$ , was established by Bochner and Montgomery[1] for finite dimensional manifolds, and extended to Banach manifolds by Dorroh[2]. Dorroh then noted that there exist  $C^r$  charts in which the generator appears to be  $C^r$  (at nonsingular points there exist  $C^r$  charts, "flow-boxes", in which it appears analytic); in a Banach space one may conjugate to a flow which actually has a  $C^r$  generator. We remark that the existence of a  $C^r$  atlas for  $M$  in which a vector field  $X$  appears  $C^r$  does not imply the existence of any  $C^{r+1}$  differentiable structure for  $M$  making  $X$  actually be  $C^r$ . In general, a  $C^{r-1}$  vector field is not even conjugate by a homeomorphism to a  $C^r$  vector field, as shown by Harrison[4]. We will show that Dorroh's result can be extended to manifolds ("globalized"), also to actions of abelian Lie groups (" $n$ -flows"), and give some applications to dynamical systems theory and differential topology. In particular, we show every  $C^r$  foliation is  $C^r$ -equivalent to another with  $C^\infty$  leaves, and that the tangent plane fields to those leaves form a  $C^r$  subbundle of the manifold's tangent bundle. These results form a part of the author's doctoral thesis; he thanks C. Pugh and M. Hirsch for advice and inspiration.

Our notation follows that of Hirsch[5], which is a general background reference. The  $C^r$  topology for maps is discussed in his chapter two; two flows  $\phi, \psi$  are  $C^r$  close if the maps  $\phi_t, \psi_t$  are, for all  $|t| \leq 1$  (we freely write  $\phi(t, p) = \phi_t(p) = \phi_p(t)$ , as convenient). Unless stated otherwise, norms and convergence of functions refers to the strong topology, and manifolds will be assumed to be  $C^\infty$  and finite dimensional. A dot is used to denote differentiation with respect to the time variable,  $t$  or  $s$ , and  $D$  denotes differentiation with respect to the space variable,  $x$  or  $p$ .

## §1. FLOWS

We consider  $n$ -parameter families of mappings, or " $n$ -flows",  $\phi(s, p)$  for  $s \in \mathbb{R}^n$  and  $p \in M$ . Such an action  $\phi$  is equivalent to  $n$  commuting flows  $\phi^1, \dots, \phi^n$ , defined by  $\phi((t_1, \dots, t_n), p) = \phi_{t_1}^1 \circ \dots \circ \phi_{t_n}^n(p)$ . We say that the vector fields  $X^i$  generating  $\{\phi^i\}$  collectively generate  $\phi$ ; note that  $[X^i, X^j] = 0$  and  $D\phi_t^i \cdot X^j(p) = X^j(\phi_t^i(p))$ .

**LEMMA 1.** *If  $X$  is a vector field generating a  $C^r$  flow and  $\lambda$  is strictly positive  $C^r$  function, then  $\tilde{X} = \lambda X$  also generates a  $C^r$  flow.*

*Proof.* This follows from the observation that the  $\tilde{X}$ -flow,  $\tilde{\phi}$ , is a reparametrization of the  $X$ -flow,  $\phi$ , i.e.  $\tilde{\phi}(t, p) = \phi(\tau(t, p), p)$ . Differentiating with respect to  $t$  gives  $\lambda(\phi(\tau, p))X(\phi, p) = \dot{\tau}(t, p)X(\phi(\tau, p))$ . If  $\tau$  is a solution of the (parametrized) ODE  $\dot{\tau}(t, p) = \lambda \circ \phi(\tau(t, p), p)$ ,  $\tau(0, p) = 0$ , then  $\tau$  is  $C^r$  since  $\lambda$  and  $\phi$  are. Therefore  $\tilde{\phi}$  is also of class  $C^r$ . ■

**COROLLARY 2.** *Any local flow is equivalent via reparametrization to a flow.*

*Proof.* Take a locally finite cover  $\{U_i\}$  of  $M$  by bounded open sets, and put  $T_i = \inf\{t; \text{for some } p \in U_i, \phi_t(p) \notin U_i\}$ . Then  $T_i > 0$ ; let  $v_i = \inf\{1, \frac{1}{2}T_i\}$  and  $\eta_i = \inf\{v_j; U_i \cap U_j \neq \emptyset\}$ . Let  $\{\lambda_i\}$  be a partition of unity subordinate to  $\{U_i\}$ ;  $\lambda(p) = \sum \eta_i \lambda_i(p)$  is strictly positive and  $|\lambda|_{C^0, U_i} = \sup\{|\lambda(p)|; p \in U_i\} \leq v_i$ . Now  $\tilde{\phi}$ , the flow of  $\tilde{X} = \lambda X$ , has minimum time of existence at least one, at every point— $\dot{\tau} \leq \lambda \leq v_i$  implies  $\tau(t) \leq v_i t$ , so  $\tau(1) \leq v_i < T_i$  and  $\tilde{\phi}(1, p) = \phi(\tau(1, p), p)$  is in  $U_i$ ,  $p \in U_i$  and  $U_i \cap U_j \neq \emptyset$ . Putting these orbit segments together, one sees that the minimum time of existence is in fact infinite, i.e.  $\tilde{\phi}$  is a complete flow. ■

More is true: if  $\phi$ ,  $\tilde{\phi}$ ,  $X$  and  $\tilde{X}$  are as above, and  $\tilde{\phi}$  is conjugate by  $g$  to a flow  $\tilde{\psi}$  with a  $C^r$  generator  $\tilde{Y}$ , then the local flow  $\psi = g\phi g^{-1}$  is also  $C^r$  generated, by  $Y = (1/\lambda \circ g^{-1})\tilde{Y}$ . We may therefore assume that the domain of  $\phi$  is  $\mathbb{R}^n \times M$ , noting that the lemma may be applied to  $X^1, \dots, X^n$  simultaneously.

We next note that Dorroh's observation can be "relativized" and extended to  $n$ -flows.

**LEMMA 3.** *Let  $\phi$  be a  $C^r$   $n$ -flow defined on an open subset  $A$  of an Euclidian space. Suppose the generators  $X^1, \dots, X^n$  of  $\phi^1, \dots, \phi^n$  are  $C^r$  on  $U \subseteq A$ , and  $U \supseteq cl(V)$ ,  $V \supseteq cl(W)$ . Let  $\gamma$  be a  $C^1$ -small  $C^{r+1}$  function vanishing only on  $W$  and constant on  $M \setminus V$ , let  $h(p) = p$  if  $p \in W$  and otherwise*

$$h(p) = (\gamma(p))^{-n} \int_0^{\gamma(p)} \cdots \int_0^{\gamma(p)} \phi(s, p) ds.$$

Then  $h$  is a  $C^r$  diffeomorphism, the identity on  $W$ , and  $C^{-1}$ -near the identity on all of  $A$ ;  $h$  conjugates  $\phi$  to a  $C^r$   $n$ -flow  $\psi$  which is  $C^r$ -generated.

*Proof.* Henceforth we take  $n = 2$ , for ease of exposition; the extension to arbitrary  $n$  is routine. We show  $h$  is a  $C^r$  diffeomorphism by showing it is close to the identity in the strong  $C^1$  topology and invoking the global inverse function theorem. If the space were compact and the "averaging interval"  $[0, c]$  constant, this would be simple:

$$\begin{aligned} \|h - id\|_{C^r} &= \left\| \frac{1}{c^2} \int_0^c \int_0^c \phi(s_1, s_2, \cdot) ds_1 ds_2 - id \right\|_{C^r} \\ &\leq \frac{1}{c^2} \int_0^c \int_0^c \|\phi_s - id\|_{C^r} ds \leq \sup_{s_i \in [0, c]} \|\phi_s - id\|_{C^r}, \text{ and } \phi_s \end{aligned}$$

would converge uniformly to the identity as  $s \rightarrow 0$ . We handle the lack of compactness by reparametrizing, using the preceding lemma. The replacement of the constant  $c$  by variable  $\gamma$  requires a careful estimate (the first inequality above is too simple, and the complexity of the proper formula for the derivatives increases with  $r$ ).

Let  $\{A_j\}$  be a family of compact subsets whose interiors form a locally finite cover of  $A$ , and set  $B_j = \phi([0, 1]^2 \times A_j)$ , so  $\{B_j\}$  also forms a locally finite cover.

$$\begin{aligned}
|\tilde{h}(p) - p| &\leq \frac{1}{\gamma^2} \int_0^\gamma \int_0^\gamma |\tilde{\phi}(s_1, s_2, p) - p| \, ds_1 \, ds_2 \\
&= \sup_{0 \leq s_i \leq \gamma} |\tilde{\phi}(s, p) - p| \\
&\leq \sup_{0 \leq s_i < \gamma} |s_1 \tilde{x}^1(\tilde{\phi}(s, p)) + s_2 \tilde{x}^2(\tilde{\phi}(s, p))| \\
&< 2 \sup_i \sup_{0 < s_i < \gamma} |\lambda(\phi(s, p)) \cdot x^i(\phi(s, p))|.
\end{aligned}$$

Hence  $\|\tilde{h} - id\|_{C^0, A_j} < \epsilon_j$  if  $\|\lambda\|_{C^0, B_j} < \frac{1}{2}\epsilon_j/\|X^i\|_{C^0, B_j}$ .

We next wish to estimate  $\|D\tilde{h} - I\|_{C^0}$ .

$$\begin{aligned}
D\tilde{h}(p) \cdot v &= \frac{1}{\gamma^2} \int_0^\gamma \int_0^\gamma D\tilde{\phi}_p(s_1, s_2) \, ds_1 \, ds_2 \cdot v \\
&\quad + D\gamma(p) \cdot v \left\{ \frac{\frac{1}{\gamma} \int_0^\gamma \tilde{\phi}_p(s_1, \gamma) \, ds_1 + \frac{1}{\gamma} \int_0^\gamma \tilde{\phi}_p(\gamma, s_2) \, ds_2 - 2\tilde{h}(p)}{\gamma} \right\}
\end{aligned}$$

The first term on the right approaches  $v$ , as  $\gamma \rightarrow 0(C^0)$ , for

$$\begin{aligned}
\left\| \frac{1}{\gamma^2} \int_0^\gamma \int_0^\gamma D\tilde{\phi}(s_1, s_2, \cdot) \, ds_1 \, ds_2 - I \right\|_{C^0, A_j} &\leq \sup\{\|D\tilde{\phi}_p(s_1, s_2) - I_p\|; s_i \in [0, \gamma(p)], p \in A_j\} \\
&= \sup\{\|D\phi_p(t_1, t_2) - I_p\|; t_i \in [0, \tau(\gamma(p))], p \in A_j\}
\end{aligned}$$

(note that  $\tau$  is monotone, for fixed  $p$ ). Since  $A_j$  is compact, for each  $\epsilon > 0$  there exists  $\delta$  such that  $0 < |t| < \delta$  implies  $\|\phi_t - id\|_{C^1, A_j} < \epsilon$ . Likewise, there exists  $v$  such that  $\lambda < v_j$  on  $B_j$  implies  $|\tau_j(s_i, p)| < \delta$ , for  $p \in A_j$  and  $s_i \leq 1$ . Finally, given any sequence of positive numbers  $v_p$ , a smooth positive function  $\lambda$  with  $\lambda < v_j$  on  $B_j$  may be constructed as in the proof of the corollary above. Hence the first term of  $D\tilde{h}$  is  $C^1$ -close to the identity.

We must now show that the remaining term is close to zero (in the strong topology); since we can take  $\gamma$   $C^1$ -close to zero, we need only show that the term in brackets is bounded. We suppress the tildas, for the moment.

$$\begin{aligned}
&\frac{1}{\gamma} \left[ \frac{1}{\gamma^2} \int_0^\gamma \int_0^\gamma [\phi_p(s_1, \gamma) + \phi_p(\gamma, s_2) - 2\phi_p(s_1, s_2)] \, ds_1 \, ds_2 \right] \\
&= \frac{1}{\gamma} \left[ \frac{1}{\gamma^2} \int_0^\gamma \int_0^\gamma (\phi_p(s_1, \gamma) - \phi_p(s_1, s_2)) + (\phi_p(\gamma, s_2) - \phi_p(s_1, s_2)) \, ds_1 \, ds_2 \right] \\
&= \frac{1}{\gamma^3} \int_0^\gamma \int_0^\gamma \left\{ \int_{s_2}^\gamma X^2(\phi_t^1 \phi_{s_1}^1(p)) \, dt + \int_{s_1}^\gamma X^1(\phi_t^1 \phi_{s_2}^2(p)) \, dt \right\} \, ds_1 \, ds_2.
\end{aligned}$$

Hence the term in question is bounded on  $A_j$  by

$$\|\tilde{X}^1\|_{C^0, B_j} + \|\tilde{X}^2\|_{C^0, B_j} = \|\lambda\|_{C^0, B_j} [\|X^1\|_{C^0, B_j} + \|X^2\|_{C^0, B_j}]$$

$< 2\epsilon_j$ . Therefore  $\gamma \rightarrow 0(C^1)$  implies  $D\tilde{h} \rightarrow I(C^0)$ , so  $\tilde{h} \rightarrow id(C^1)$  and  $\tilde{h}$  is a  $C^r$  diffeomorphism.

Finally,  $\tilde{h}$  conjugates  $\tilde{X}^i$  to a  $C^r$  vector field. If  $\tilde{h}$  is sufficiently close to the identity that

$\tilde{h}^{-1}(M \setminus U) \subseteq M \setminus V$ , the term  $D\gamma(\tilde{h}^{-1}(p)) \cdot \tilde{X}^i(\tilde{h}^{-1}(p))$  vanishes unless  $\tilde{X}^i$  is  $C^r$ . The other term is similar to the flow case ([1, 2]):

$$\begin{aligned} \frac{1}{\gamma^2} \int_0^\gamma \int_0^\gamma D\phi_{s_1}^1 [\circ D\phi_{s_2}^2(p)] \circ X^1(p) \, ds_1 \, ds_2 &= \frac{1}{\gamma^2} \int_0^\gamma \int_0^\gamma X^1(\phi_{s_1}^1 \circ \phi_{s_2}^2(p)) \, ds_1 \, ds_2 \\ &= \frac{1}{\gamma^2} \int_0^\gamma (\phi_{s_1}^1 \phi_{s_2}^1(p) - \phi_{s_2}^2(p)) \, ds_2, \end{aligned}$$

and similarly for  $X^2$ ; this is  $C^r$ , if  $\phi$  is.  $\blacksquare$

Combining the above result with well-known facts about parameter dependence, one has that if  $\phi(q) = \phi(s, p, q)$  is a family of  $C^k$  flows, depending on a parameter  $q$  in a  $C^j$  fashion, then there is a  $C^j$  family of  $C^k$  diffeomorphisms  $g(p, q)$  conjugating  $\phi(q)$  to a parametrized  $n$ -flow  $\psi(q)$  generated by a  $C^j$  family of  $C^k$  vector fields. All of these objects are jointly  $C^j$ , if  $k \geq j$  [3]. The conjugacy may be taken uniformly  $C^1$ -close to the identity, as above (actually, it may be taken  $C^j$ -close to the identity, at least for  $j \leq 3$ , and  $C^k$ -close for each fixed  $q$ , but we will not need this fact).

**THEOREM A.** *Any  $C^r$  local  $n$ -flow  $\phi$  is  $C^r$ -conjugate to an action generated by  $n$  commuting  $C^r$  vector fields. If  $U, W \subseteq M$  with  $cl(W) \cap U$  and each  $X^i|_U$  of class  $C^r$  already, the conjugacy may be taken to be the identity map on  $W$ ; it is  $C^1$ -close to the identity, in any case.*

*Proof.* We construct a sequence of maps supported in charts using the lemma, and patch them together to give the desired conjugacy. Let  $rD^m = \{x \in \mathbb{R}^m; |x| \leq r\}$  and  $D^m$  the unit disk of  $\mathbb{R}^m$ . Suppose  $\{\alpha_j, U_j\}$  is a locally finite atlas of  $M$ ,  $\alpha_j(U_j) = D^m$ , and  $W_j = \alpha_j^{-1}(\frac{1}{2}D^m)$  such that  $\{W_j\}$  covers  $M$ .

At the first step, we use a bump function  $\gamma_1$  supported in  $0.9 D^m$  and constant on  $0.8 D^m$  to construct the map  $g_1 = \alpha_1^{-1} \circ h \circ \alpha_1$  supported in  $U_1$ . This map extends to the identity on the rest of  $M$ , and conjugates  $\phi$  to a flow  $\psi^{(1)}$  which is  $C^r$ -generated on a neighborhood  $V_1$  of  $W_1$ .

At the  $j$ th step, we require that  $\gamma_j$  be supported in  $0.9 D^m \setminus \alpha_j(\cup_{i < j} W_i)$  and be constant on a neighborhood of that part of  $0.8 D^m$  in which  $\alpha_j \circ \psi^{(j-1)} \circ \alpha_j^{-1}$  is not already  $C^r$ -generated,  $0.8 D^m \setminus \alpha_j(V_{j-1})$ . We obtain  $\psi^{(j)}$  agreeing with  $\phi^{(j-1)}$  on  $W_i$ , agreeing with  $\phi$  on  $M \setminus \cup_{i \leq j} U_i$ , and  $C^r$ -generated on a neighborhood  $V_j$  of  $\cup_{i \leq j} W_i$ .

Since the cover is locally finite, the  $\psi^{(j)}$  eventually stabilize (for finite time intervals) at every point;  $g = \lim_j g_j$  and  $\psi = \lim_j \psi^{(j)}$ .  $\blacksquare$

The conjugacies constructed above leave the fixed points of the flow fixed, and also the periods of the periodic points. Hence abelian Lie group (i.e.  $\mathbb{R}^p \times T^q$ ) actions may be regarded as  $\mathbb{R}^n$  actions.

## §2. FOLIATIONS

We will use theorem A to prove an analogous result for foliations, based on the observation that within any single foliation chart there is a  $C^r$  local action subordinate to the foliation (i.e. orbits = leaves). For example, if  $\alpha: U \rightarrow D^m$  is such a chart, so that in  $U$  the leaves are  $\alpha^{-1}(D^k \times \{y\})$ , for fixed  $y \in \mathbb{R}^{m-k}$ , then for  $s \in \mathbb{R}^k$ ,  $p \in U$ , define  $\phi(s, p) = \alpha^{-1}(\alpha(p) + s)$  whenever  $(\alpha(p) + s) \in D^m$ . By cutting  $U$  down to  $\alpha^{-1}(0.9 D^m)$ , we may assume a uniform minimum time of existence for  $\phi$  on  $U$  and apply the preceding result.

**THEOREM B.** *If  $F$  is a  $k$ -dimensional  $C^r$  foliation of a differentiable manifold  $M$ , there is a  $C^r$  diffeomorphism of  $M$ ,  $C^1$ -close to the identity, conjugating  $F$  to a foliation generated by a  $C^r$   $k$ -plane field, with  $C^\infty$  leaves.*

*Proof.* Let  $\{\alpha_j, U_j\}$  be a locally finite  $C^r$  atlas of  $M$  by foliation charts inducing subordinate  $C^r$  local  $\mathbb{R}^k$ -actions, as above. Assume  $M \subseteq \bigcup W_j$ , where  $W_j \subseteq U_j$ .

In a single chart,  $U_1$ , we may take a  $C^{r+1}$  bump function,  $\gamma(p)$ , vanishing near the boundary of  $U_1$  and equal to a small constant on a neighborhood  $\tilde{U}_1$  of  $cl(W_1)$ . Defining  $g_1 \in \text{Diff}^r(M)$  as in the proof of theorem A (using  $\alpha$  to push  $\gamma$  over to the linear space) we obtain  $k$  vector fields

$$Y^i(p) = Dg_1 \circ \frac{\partial \phi}{\partial s_i} \Big|_{s=0} (g_1^{-1}(p)),$$

$C^r$  on a neighborhood  $V_1$  of  $W_1$ . They are independent by subordination, so give a  $k$ -dimensional plane field, tangent to  $F_1 = g_{1*}F$  (i.e. the new  $F_1$ -leaf through  $p$  is the image by  $g_1$  of the old  $F$ -leaf through  $g_1^{-1}(p)$ ), which is a  $C^r$  subbundle on  $V_1$  and is the tangent field to  $F$  near the boundary of  $U_1$ . By extending  $g_1$  to be the identity map on  $M \setminus U_1$ ,  $F_1 = F$  on  $M \setminus U_1$ , and is  $C^r$ -generated to  $V_1$ . It only remains to show how to patch these together.

At the  $j$ th step,  $j > 1$ , take  $\gamma$  with support in  $U_j \setminus \bigcup_{i < j} W_i$ , vanishing near the boundary of  $U_j$  and equal to a constant on  $\tilde{U}_j \setminus V_{j-1}$ , so small that, defining  $g_j$  as before,  $cl(g_j^{-1}(W_j)) \subseteq U_j$ . Then  $g_j$  is the identity on  $\bigcup_{i < j} W_i$  and near the boundary of  $U_j$ , and produces  $k$  independent vector fields which are  $C^r$  on a neighborhood  $V^j$  of  $W_j$  as well as on  $V^{j-1} = g_j(V_{j-1})$ , a neighborhood of  $\bigcup_{i < j} W_i$ . Set  $F_j = g_{j*}F_{j-1}$ , so  $F_j = F_{j-1}$  on  $\bigcup_{i < j} W_i$  and  $M \setminus U_j$ , and  $F_j$  has a  $C^r$  tangent plane field on  $V_j = V^j \cup V^{j-1}$ . The atlas is locally finite, so this eventually stabilized at each point; set  $g^1 = \lim_j (g_j \circ \dots \circ g_1)$ . Then  $G_1 = g^1_* F$  has a  $C^r$  tangent plane field on all of  $M$ .

The leaves of  $G$  are (a  $C^r$  family of)  $C^{r+1}$  immersed submanifolds of  $M$ . We want to iterate the process of "tangentially smoothing" the leaves, obtaining  $C^r$  diffeomorphisms  $g^k$  making the leaves  $C^{r+k}$  submanifolds. Since  $h_*X = (1/\gamma)(\phi_\gamma - id)$  is  $C^{r-1}$ -close to  $X$  by definition (refer to lemma 3, for  $h$ ), and if  $X$  is  $C^j$  on a leaf then they are  $C^j$ -close in that leaf, this can be done (see the remark following Lemma 3). We obtain, in the limit, a  $C^r$  foliation  $g_*F$  with  $C^\infty$  leaves; that  $g$  may be taken to be  $C^r$  follows from section two of chapter two of [5]. ■

### §3. APPLICATIONS

Theorem A implies the linear space of  $C^r$  vector fields and the nonlinear space of  $C^r$  flows are in some sense equivalent, that it is possible to "translate" theorems from one category to the other. In this section we give some of these applications.

**COROLLARY 4.** *If  $M$  is a submanifold of  $N$ , and  $\phi$  is a  $C^r$  flow on  $M$ , there is a  $C^r$  flow  $\Phi$  on  $N$  extending  $\phi$ .*

*Remark.* Extending a family of diffeomorphisms is of course no problem; it is the group property of a flow which makes this nontrivial.

*Proof.* We translate the problem to the space of vector fields, and use Whitney's extension theorem [11]. Let  $g \in \text{Diff}^r(M)$  be as in theorem A, such that  $\psi = g\phi g^{-1}$  has a  $C^r$  generator  $Y$ . By Whitney's theorem there exist  $C^r$  extensions  $G$  and  $\tilde{Y}$  to  $N$ ; let  $\Psi$  be the flow generated by  $\tilde{Y}$ . Then  $\Psi$  is  $C^r$ , as is  $\Phi = G^{-1}\Psi G$ , and  $\Phi|_M = g^{-1}g\phi g^{-1}g = \phi$ . ■

**COROLLARY 5.** *Any  $C^r$  diffeomorphism  $f$  of  $M$  can be embedded as the Poincaré first-return map for a  $C^r$  vector field  $X$  of  $M^* = (M \times I)/R$ , where  $I$  is the unit interval  $[0, 1]$  and  $R$  is the relation  $\{((x, 1), (f(x), 0))\}$ .*

*Proof.* It is well known that  $f$  embeds as the time-one map of the  $C'$  flow  $\phi$  on  $M^*$  generated by the unit vector field along the interval factor,  $(\partial/\partial t) = (0, 1)$  in  $M \times I$  "coordinates" (see Smale[10]). The manifold  $M^*$  has a compatible  $C^\infty$  structure such that  $\phi$  is still  $C'$ , but its generating vector field remains only  $C'^{-1}$ . Theorem A shows there is a  $C'$  diffeomorphism  $g$  of  $M$  conjugating  $\phi$  to a flow  $\psi$  with a  $C'$  vector field. We may assume the averaging interval is  $[0, c]$ ,  $c$  a small constant.

At this point we have only that  $f$  is conjugate by  $g$  to the first return map of  $\psi$  (on a cross section  $M' = (M \times \{\frac{1}{2}\})$ , say). However, we may require that  $M^*$  be embedded so that the  $t$ -fibers in  $M \times (\frac{1}{2} - 2c, \frac{1}{2} + 2c)$  are actually straight line segments, and so  $g = h = id$  on  $M \times (\frac{1}{2} - c, \frac{1}{2} + c)$ . We are thus assured that the first-return map  $\psi_{\tau(p, 1/2)} = \phi_1 = f$  is unchanged on  $M \times \{\frac{1}{2}\}$ . The return-time map  $\tau(p, \frac{1}{2})$  might not remain identically equal to one, but this may be arranged by reparametrizing. ■

Another application is the existence of  $C^1$  Anosov vector fields whose stable-unstable manifold foliations are not absolutely continuous. The existence of such  $C^1$  diffeomorphisms was shown by Robinson and Young[9], and the vector field case becomes an easy corollary. Note that it is crucial here that the conjugacy is itself  $C^1$ , and so preserves absolute continuity.

It is also possible to obtain generic properties for vector fields from the corresponding properties for flows, and vice versa. This involves the actual form of the conjugacy constructed above, and goes as follows.

**COROLLARY 6.** *The  $C'$  Closing Lemma is true for vector fields if and only if it is true for flows. In particular, the Closing Lemma holds for  $C^1$  vector fields and for volume preserving  $C^1$  flows.*

*Proof.* The Closing Lemma is proved within a single chart, i.e. Euclidian space. Let  $X$  be a  $C'$  vector field with flow  $\phi$ , and  $h_1$  the associated finite averaging map, as in Lemma 3, but with the averaging interval constant. Suppose  $\psi$  is a flow  $C'$ -close to  $\phi$ , with associated map  $h_2$ , where the averaging interval has the same length  $c$  for  $h_2$  as for  $h_1$ . Then  $h_2$  is  $C'$  close to  $h_1$ , and although  $Y = \dot{\psi}$  need not be  $C'$ ,  $\tilde{Y} = Dh_2 \cdot Y \cdot h_2^{-1} = (1/c)(\psi_c - id)$  is, and is indeed  $C'$ -close to  $\tilde{X} = (1/c)(\phi_c - id)$ .

As  $c \rightarrow 0$ ,  $\tilde{X} \rightarrow X(C'^{-1})$ , by the definition of  $X$ ; we want to show  $C'$  convergence, if  $X$  is  $C'$ . Note that  $D\tilde{X} = (1/c)(D\phi_c - I)$ , and  $DX = (D\phi_t) \cdot D\phi_{-t}$ . Thus it will be enough to show that  $(1/c)(D\phi_{c+t} - D\phi_t) - (D\phi_t)$  is small, in the  $C'^{-1}$  sense, for small  $c$  and  $t$ . Since  $X$  is  $C'$ ,  $\phi(t, p)$  is  $C'^{+1/r}$  (i.e.,  $C'^{+1}$  in  $t$ , and  $C'$  in  $p$ ),  $D\phi(t, p)$  is  $C'^{r-1}$ , while  $D\phi(t, p)$  is  $C'^{-1}$ ; but then  $(1/c)(D\phi_{c+t} - D\phi_t) \rightarrow D\phi_t$  in the  $C'^{-1}$  sense, as  $c \rightarrow 0$ , which implies that  $\tilde{X}$  is  $C'$ -close to  $X$ , for  $c$  sufficiently small. From this it follows that  $\tilde{X} = Dh_1 \circ X \circ h_1^{-1}$  is  $C'$ -close to  $X$ , as well as to  $\tilde{Y}$ .

We have  $\psi$  near  $\phi$  leading to  $\tilde{Y}$  near  $\tilde{X}$  near  $X$ , so that  $C'$  approximation of  $\phi$  by another flow (such that the trajectory through a given non-wandering point is periodic, say) is equivalent to  $C'$  approximation of  $X$  by another vector field (with the same property).

This fills a gap in the published proof of the Closing Lemma[7], in which the result is claimed for  $C^1$  vector fields but only proved for  $C^1$  flows. It also permits a slight extension of the results of [8]: the Closing Lemma is true for  $C^1$  volume-preserving flows, since it is true for the corresponding vector fields. Since the Closing Lemma is proved within a single chart, it suffices to consider a flow preserving Lebesgue measure on the unit cube in  $R^n$ , by a result of Moser[6]. The averaging map  $h$  is not necessarily volume-

preserving; nevertheless,  $\psi = h\phi h^{-1}$  is, if  $\phi$  is:

$$\begin{aligned}\frac{d}{dt}(\det D\psi_t) &= \operatorname{tr}(\dot{\psi}_t) \det(D\psi_0) \\ &= \operatorname{tr}(Dh \circ \dot{\phi}_t \circ h^{-1}) \\ &= \operatorname{tr}\left(\frac{1}{c} \int_0^c D\phi_s \circ \dot{\phi}_t \circ h^{-1} ds\right) \\ &= \frac{1}{c} \int_0^c \operatorname{tr}(\dot{\phi}_t \circ \phi_s \circ h^{-1}) ds = 0.\end{aligned}$$

Therefore  $\det(D\psi_t) = \det(D\psi_0) = 1$ . Hence any volume-preserving  $C^r$  flow is locally conjugate to the flow of a divergence-free  $C^r$  vector field, and the Closing Lemma is true for  $C^1$  volume-preserving flows. ■

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